# Zooming into Vlasov–Poisson using a Characteristic Mapping Method

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## Research aims

Study of **plasmas** on microscopic levels with long-range interaction:

- physical understanding of **fine-scale properties**
- development of **efficient simulation** of high dimensional nonlinear systems
- methodical developments in **model order reduction** and numerical methods for PDEs

## Vlasov–Poisson

A **kinetic model** as first-principle physics for noncollisional plasmas

$$\partial_t f + v\partial_x f + E\partial_v f = 0 \tag{1}$$

- $f(x,v,t)$ one-particle **probability density function (PDF)**
- $f\,\mathrm{d}x\mathrm{d}v$ is the probability to find a praticle with the certain velocity $v$ and position $x$ at time $t$
- $E(x,t) = -\partial_x \phi(x,t)$ **electric field** determined by

$$\partial_x E = \rho - \rho_0 \quad \text{(Gauss Law)} \tag{2}$$

where $\rho(x,t) = \int f(x,v,t)\mathrm{d}v$ is the density

### Properties of the model

- **conservation** of

$$\mathcal{M}(t) = \int\int f(x,v,t)\,\mathrm{d}v\mathrm{d}x \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}(t) = 0 \qquad \text{(mass)}$$

$$\mathcal{P}(t) = \int\int f(x,v,t)v\,\mathrm{d}v\mathrm{d}x \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(t) = 0 \qquad \text{(momentum)}$$

$$\mathcal{E}(t) = \mathcal{E}_{\text{kin}}(t) + \mathcal{E}_{\text{pot}}(t) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = 0 \qquad \text{(energy)}$$

where

$$\mathcal{E}_{\text{kin}}(t) = \int\int f(x,v,t)|v|^2\,\mathrm{d}v\mathrm{d}x \qquad \mathcal{E}_{\text{pot}}(t) = \int |E(x,t)|^2\,\mathrm{d}x \tag{3}$$

- **divergence free** velocity field $\boldsymbol{u} = (v, E(x,t))$

$$\nabla\cdot\boldsymbol{u} = \partial_x v + \partial_v E(x,t) = 0 \tag{4}$$

## Characteristic Mapping Method (CMM) [1, 2]

The CMM considers a non-linear advection equation

$$\begin{cases} \partial_t\theta + \boldsymbol{u}\cdot\nabla\theta = 0 & \text{for } (\boldsymbol{x},t)\in\Omega\times\mathbb{R}_+ \\ \theta(\boldsymbol{x},0) = \theta_0(\boldsymbol{x}) & \boldsymbol{x}\in\Omega \end{cases} \tag{5}$$

with velocity field $\boldsymbol{u}\colon \mathbb{R}\times\Omega\times\mathbb{R}_+\to\mathbb{R}^d; (\theta,\boldsymbol{x},t)\mapsto\boldsymbol{u}(\theta,\boldsymbol{x},t)$ that may depend on the advected state $\theta\colon(\boldsymbol{x},t)\colon\Omega\times\mathbb{R}_+\to\mathbb{R}$ itself. If we follow the points $(\boldsymbol{x},t) = (\boldsymbol{\gamma}(t),t)$ that satisfy the ODE:

$$\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t} = \boldsymbol{u} \quad \text{with} \quad \boldsymbol{\gamma}(0) = \boldsymbol{x}\in\Omega \tag{6}$$

we have:

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}(\boldsymbol{\gamma}(t),t) = \partial_t\theta + \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}\cdot\nabla\theta = 0\,. \tag{7}$$

This means that along these points the solution stays constant in time and can be mapped back to its initial values:

$$\theta(\boldsymbol{\gamma}(t),t) = \theta_0(\boldsymbol{x})\,. \tag{8}$$

### CMM for Vlasov–Poisson

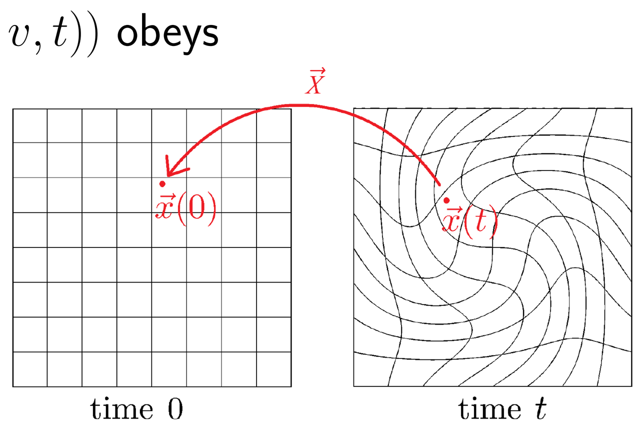
The characteristic map $\boldsymbol{X}(x,v,t) = (X(x,v,t), V(x,v,t))$ obeys

$$\partial_t\boldsymbol{X} + \boldsymbol{u}\cdot\nabla\boldsymbol{X} = 0 \quad \text{where} \quad \boldsymbol{u} = (v, \partial_x\phi) \tag{9}$$

$$\partial_{xx}\phi = l\int f\,\mathrm{d}v - 1 \tag{10}$$

with initial condition:

$$\boldsymbol{X}(x,v,0) = (x,v) \tag{11}$$




It relates the advection of the distribution function to its initial distribution:

$$f(x,v,t) = f_0(X(x,v,t), V(x,v,t)) \tag{12}$$

For numerical efficiency we exploit the semi-group property of the characterisitic map:

$$\boldsymbol{X}_{[\tau_{i-1},\tau_i]}\colon \begin{cases} \partial_t\boldsymbol{X} + \boldsymbol{u}\cdot\nabla\boldsymbol{X} = 0 & \text{fort } \in[\tau_{i-1},\tau_i] \\ \boldsymbol{X}(x,v,\tau_{i-1}) = (x,v) \end{cases} \tag{13}$$

Thus $\boldsymbol{X}_{[0,T]} = \boldsymbol{X}_{[0,\tau_1]}\circ\boldsymbol{X}_{[\tau_1,\tau_2]}\circ\cdots\circ\boldsymbol{X}_{[\tau_{m-2},\tau_{m-1}]}\circ\boldsymbol{X}_{[\tau_{m-1},T]}$

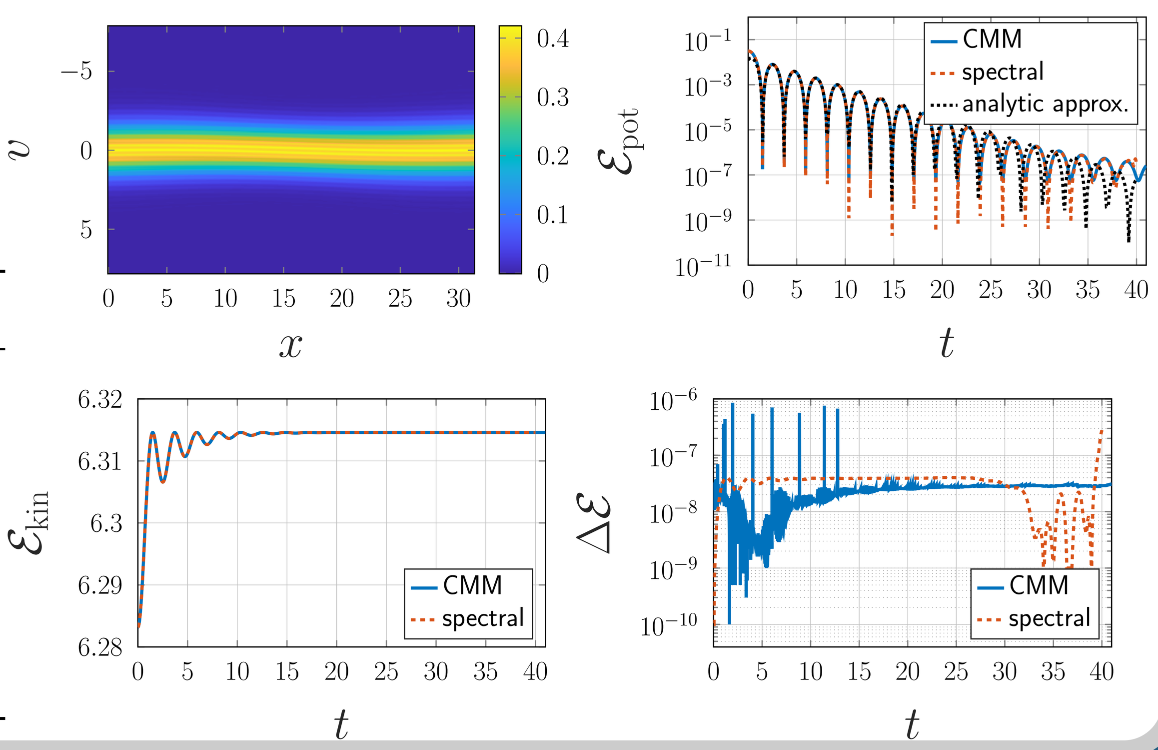
## Landau Damping

Initial condition:

$f_0(x,y) = \frac{1}{l}(1+\epsilon\cos(kx))\frac{1}{\sqrt{2\pi}}e^{-v^2/2}$

Simulation parameters

| name | value |
|---|---|
| domain size | $L = 2\pi/k$ |
| velocity interval | $v\in[-8,8]$ |
| wavevector $k$ | 0.2 |
| $\epsilon$ | $5\times10^{-2}$ |
| $l$ | 1 |
| $T$ | 40 |



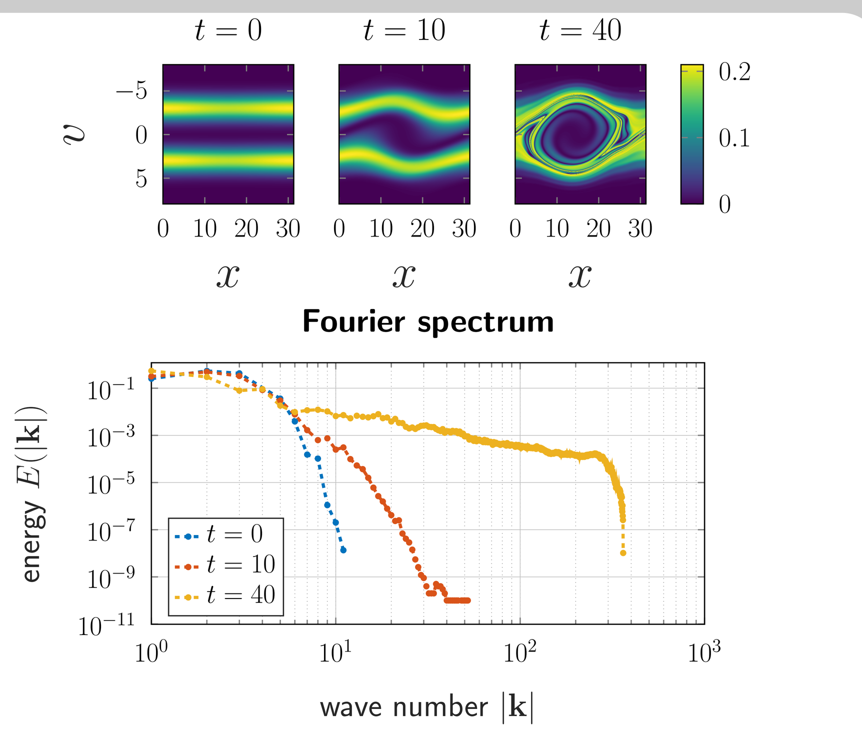

## Two-Stream Instability

Initial condition:

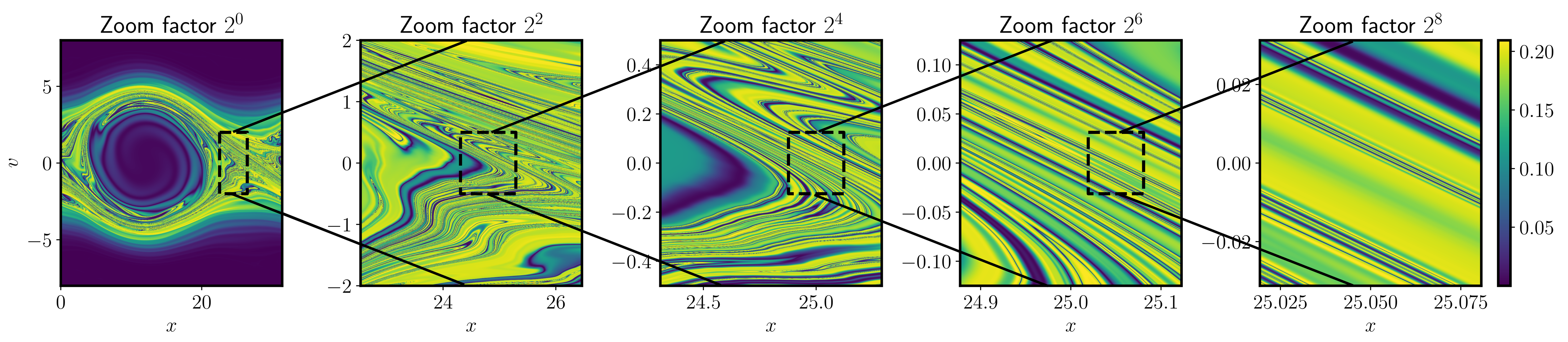
$f_0(x,y) = \frac{1}{l}(1+\epsilon\cos(kx))\frac{1}{2\sqrt{2\pi}}(e^{-(v-v_0)^2/2} + e^{-(v+v_0)^2/2})$

Simulation Parameters

| name | value |
|---|---|
| domain size | $L = 2\pi/k$ |
| velocity interval | $v\in[-2.5\pi, 2.5\pi]$ |
| wavevector $k$ | 0.2 |
| $\epsilon$ | $5\times10^{-2}$ |
| $l$ | 1 |
| $v_0$ | 3 |
| $T$ | 80 |




## Zoom into Two-Stream Instability




## References

[1] Philipp Krah, Xi-Yuan Yin, Julius Bergmann, Jean-Christophe Nave, and Kai Schneider. A characteristic mapping method for vlasov–poisson with extreme resolution properties. *to be submitted*, 2023.

[2] Xi-Yuan Yin, Olivier Mercier, Badal Yadav, Kai Schneider, and Jean-Christophe Nave. A characteristic mapping method for the two-dimensional incompressible Euler equations. *Journal of Computational Physics*, 424:109781, 2021.

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