Dynamical low rank approximation and parametric reduced order models for shallow water moment equations

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Background

Model reduction of shallow free-surface flows:

(1) Tsunamis

(2) Avalanches

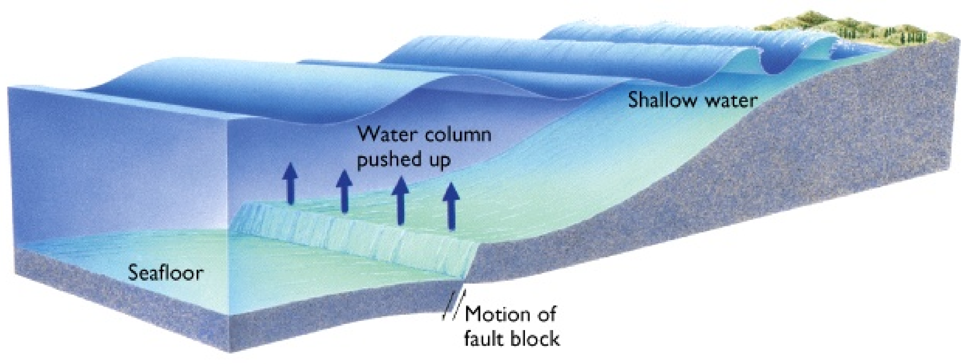
(3) Atmospheric currents

Research aims

Reduction of complexity using two methods:

- Dynamical low rank approximation (DLRA)
- Proper orthogonal decomposition (POD)

Shallow flows



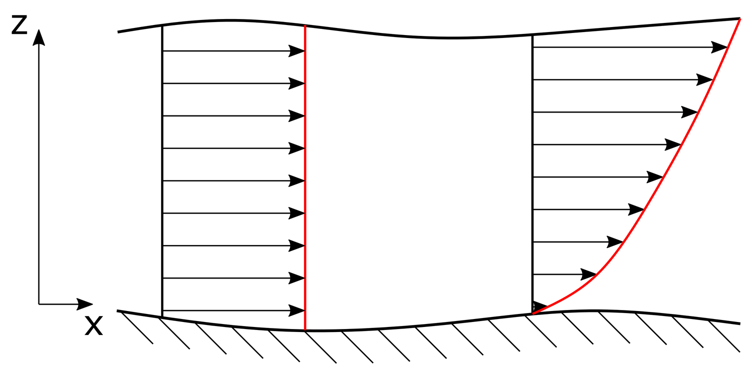
Relevant scale is the **shallowness**

$$S = \frac{\text{water height}}{\text{wave length}} = \frac{h}{\lambda} \ll 1$$

Idea: Expand unknown horizontal velocity profile in Legendre series around mean velocity

$$u(t,x,z) = u_m(t,x) + \sum_{i=1}^{M} \alpha_i(t,x)\phi_i\left(\frac{z-h_b}{h}\right)$$

Shallow Water Moment Equations [1]



Leads to **hyperbolic Shallow Water Moment Model (HSWME)** [2]

$$\partial_t \boldsymbol{u}_M + \boldsymbol{A}_M(\boldsymbol{u}_M)\partial_x \boldsymbol{u}_M = \boldsymbol{S}(\boldsymbol{u}_M), \quad \boldsymbol{u}_M = (h, hu_m, h\alpha_1, \dots, h\alpha_M)^T \in \mathbb{R}^{M+2} \quad (1)$$

$$\boldsymbol{A}_M(\boldsymbol{u}_M) = \begin{bmatrix} 0 & 1 & & & & \\ gh - u_m^2 - \frac{1}{3}\alpha_1^2 & 2u_m & \frac{2}{3}\alpha_1 & & & \\ -2u_m\alpha_1 & 2\alpha_1 & u_m & \frac{3}{5}\alpha_1 & & \\ -\frac{2}{3}\alpha_1^2 & 0 & \frac{1}{3}\alpha_1 & u_m & \ddots & \\ & & & \ddots & \ddots & \frac{N+1}{2N+1}\alpha_1 \\ & & & & \frac{N-1}{2N-1}\alpha_1 & u_m \end{bmatrix}$$

Model Order Reduction

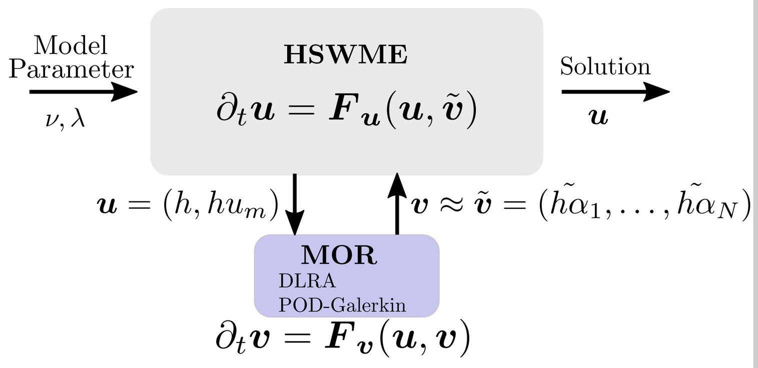
Separation of **HSWME** in:

$$\partial_t \boldsymbol{u} = \boldsymbol{F}_{\boldsymbol{u}}(\boldsymbol{u}, \boldsymbol{v}) \quad (2)$$

to describe $\boldsymbol{u} = (h, hu_m)$ and

$$\partial_t \boldsymbol{v} = \boldsymbol{F}_{\boldsymbol{v}}(\boldsymbol{u}, \boldsymbol{v}) \quad (3)$$

for the higher moments $\boldsymbol{v} = (h\alpha_1, \dots, h\alpha_M)$.



Goal: Find a cheap approximation of the moment equations 3, while keeping conservation properties for $\boldsymbol{u}$.

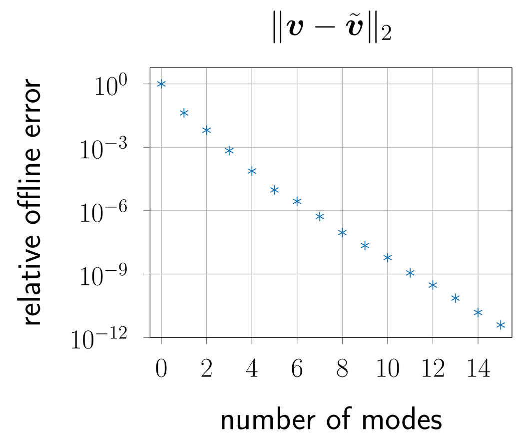
POD-Galerkin Offline-Online Procedure [4]

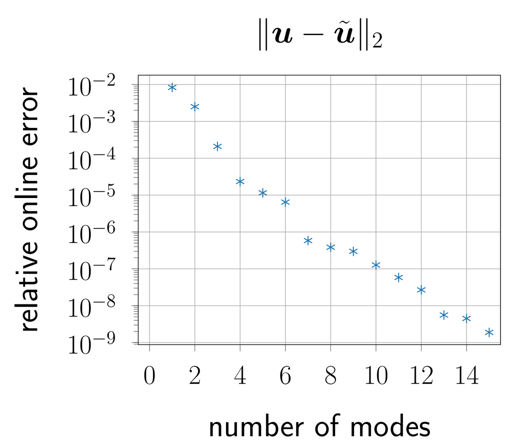
Offline

- Collect solution snapshots solving eq. (1) explicitly
  $$\mathcal{V} = \{\boldsymbol{v}(x,t_1), \dots, \boldsymbol{v}(x,t_N)\}$$
- POD approximates the higher moment vector using orthonormal basis $\{\boldsymbol{w}\}_{k=1,\dots,M}$:
  $$\boldsymbol{v}(x,t) \approx \tilde{\boldsymbol{v}}(x,t) = \sum_{k=1}^{r} \hat{\alpha}_k(x,t)\boldsymbol{w}_k \qquad r \ll M$$

Online

- Evolve eq. (2) using $\tilde{\boldsymbol{v}}$ to obtain $\boldsymbol{u}$
- Evolve dynamics on reduced $\boldsymbol{v}$-space:
  $$\partial_t \hat{\alpha}_k(x,t) = \langle \boldsymbol{w}_k, \boldsymbol{F}_{\boldsymbol{v}}(\boldsymbol{u}, \tilde{\boldsymbol{v}})\rangle \qquad k = 1, \dots, r \ll M$$



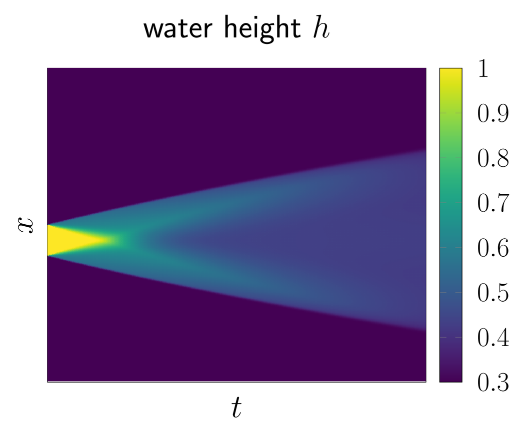


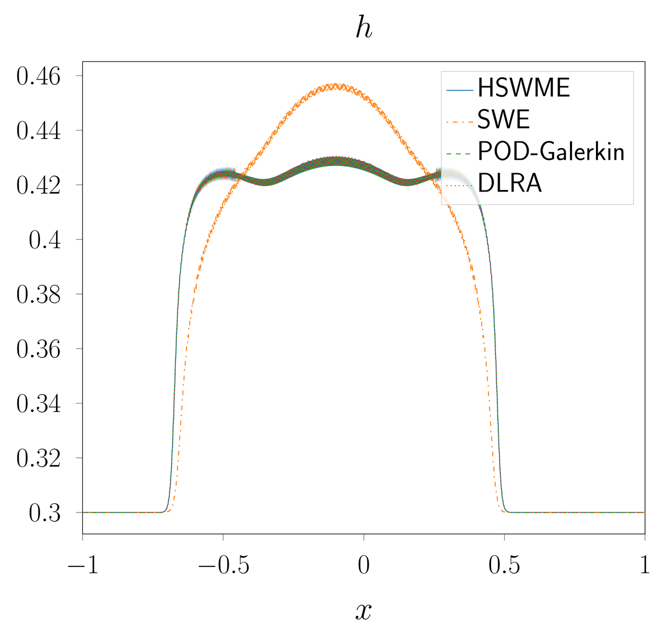
Results: POD-Galerkin vs. DLRA

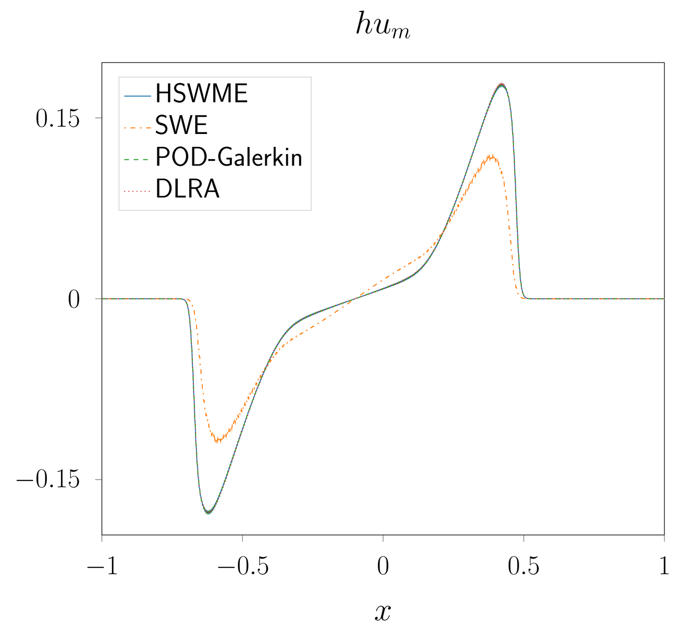
Dam-break Problem:
Simulation in $(x,t) \in [-1,1] \times [0, 02]$ with $\nu = 1, \lambda = 0.5$ initial condition:

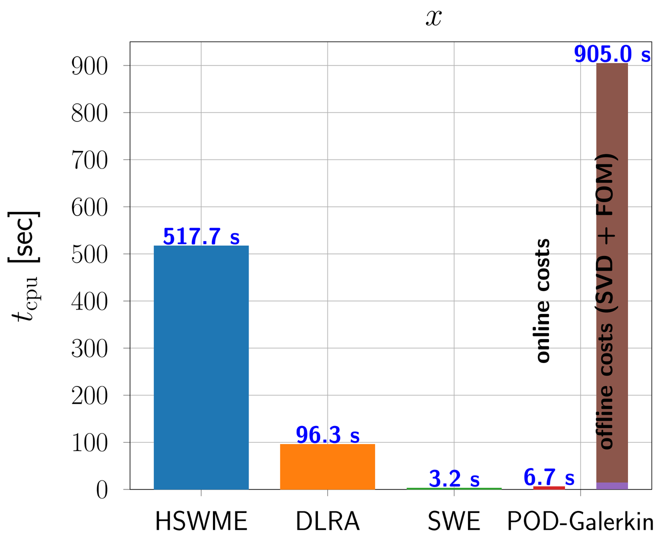
$$h(x, t=0) = 0.3 + 0.7\,\mathrm{rect}_{[-0.2,0]}(x)$$
$$u_m(x, t=0) = 0$$

Discretized with $N_x = 1000$ grid points and $M = 100$ moments.









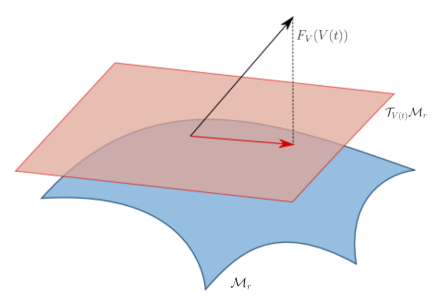
- POD $r = 3$, DLRA $r = 5$ to obtain approximately the same accuracy (rel. err. $\approx 0.5\%$) for DLRA and POD-Galerkin.
- SWE corresponds to $\alpha_i = 0$
- POD snapshot set $\mathcal{V}$ is obtained from 800 equally spaced snapshots computed with $\nu = 0.1, 10$

Dynamical low-rank approximation [5]

DLRA evolves $\mathbf{V}(t) = [h(t,x_i)\alpha_j(t,x_i)]_{ij} \in \mathbb{R}^{N_x \times M}$ on the manifold or rank r matrices $\mathcal{M}_r$

$\dot{\mathbf{V}}(t) \in \mathcal{T}_{\mathbf{V}(t)}\mathcal{M}_r$ such that $\|\dot{\mathbf{V}}(t) - \mathbf{F}_{\mathbf{V}}(\mathbf{V}(t))\| \to$ min!

where $\mathbf{F}_{\mathbf{V}}(\mathbf{V}(t)) := \boldsymbol{F}_{\boldsymbol{v}}(\boldsymbol{u}, \mathbf{V}(t))$.



The DRLA - evolution equations for $\mathbf{V}(t) = \mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top$ are:

$$\dot{\mathbf{S}}(t) = \mathbf{X}(t)^\top \mathbf{F}_{\mathbf{V}}(\mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top)\mathbf{W}(t),$$
$$\dot{\mathbf{X}}(t) = (\mathbf{I} - \mathbf{X}(t)\mathbf{X}(t)^\top)\mathbf{F}_{\mathbf{V}}(\mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top)\mathbf{W}(t)\mathbf{S}(t)^{-1},$$
$$\dot{\mathbf{W}}(t) = (\mathbf{I} - \mathbf{W}(t)\mathbf{W}(t)^\top)\mathbf{F}_{\mathbf{V}}(\mathbf{X}(t)\mathbf{S}(t)\mathbf{W}(t)^\top)^\top\mathbf{X}(t)\mathbf{S}(t)^{-\top}.$$

Use robust *basis update & Galerkin step* integrator [6] to solve in time.

Conclusion - Comparison

POD-Galerkin	DLRA
+ faster and more accurate (online phase)	– more complex and slower
– expensive offline phase	+ no initial setup or offline phase needed

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